SUMMER QUIZ 2011
SOLUTIONS – PART 3

Hard 1

Three coins are placed in a bag: one is an ordinary coin, the second has heads on both sides, and the third has tails on both sides. A coin is taken from the bag, is flipped, and comes up heads. What are the chances that the other side of the coin is heads?

Answer: 2/3.

Label the six sides of the coins H1 and T1 (for the normal coin), H2 and H3 (for the two-headed coin), and T2 and T3 (for the two-tailed coin). When a coin is drawn and flipped, each of the three heads has an equal chance of appearing. Of those three heads, two of them (H2 and H3) have a head on the other side. So, the chances of heads on the other side is 2/3.

HARD 2

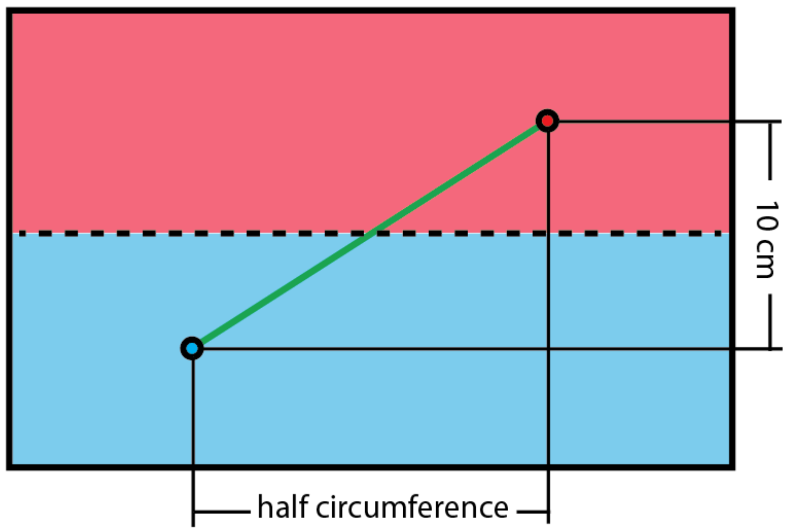
Two ants are on a cylindrical glass that is 5 centimeters in diameter. The ants are on opposite sides of the glass, 5 centimeters down from the glass's rim. If both ants are on the outside of the glass, what is the shortest distance required for one ant to crawl to the other? What if one ant is on the outside of the glass and the other is on the inside?

Answer: About 7.9 cm, and about 12.7 cm.

First consider the situation where both ants are on the outside of the glass. It is then easy to see that the shortest path is for the one ant to simply stay at the same height and walk around the glass. The length of that trip is half the circumference of the glass, which is $5\pi/2$ cm, about 7.9 cm.

Notice we can think of this path as a straight line. Imagine a piece of paper wrapped around the outside of the glass, and with the ants sitting on the paper. Unwrapping the paper and placing on a flat table, the indicated route is a straight line connecting the ants.

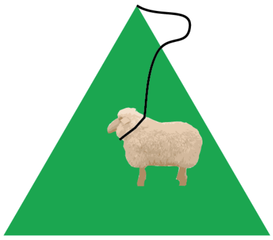
Now, consider the situation where one ant is outside the glass and the other is inside. Imagine a piece of paper folded in two, with one half wrapped around the outside of the glass, the other half around the inside, and the crease of the paper coinciding with the rim of the glass; imagine the ants sitting on the paper. Unfolding the paper, the shortest route should again be a straight line, indicated by the green line below.



Using Pythagoras's theorem, the length of the green path can be found to be $\sqrt{(5\pi/2)^2 + 10^2}$, about 12.7 cm.

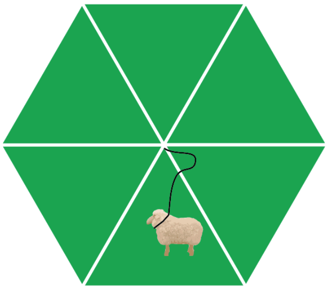
HARD 3

Fluffy the sheep is chained at one corner of an equilateral triangular field. The field has an area of 100 square meters, and Fluffy's chain is just long enough so that she can graze half the area of the field. How long is her chain?



Answer: About 9.8 metres.

Think of six fields together, forming a regular hexagon.



The hexagon has area 600 square meters, and so Fluffy's chain allows her to graze a circle of area 300 square meters. The length of Fluffy's chain is simply the radius of this circle, which is $\sqrt{300/\pi}$ metres, about 9.8 metres.

HARD 4

A room is lined with closed lockers, numbered from 1 to 100. A student enters the room and opens all the lockers. A second student enters the room and closes every second locker, all those labelled with an even number. A third student enters the room and walks up to every third locker: if it is open she closes it, and otherwise she opens it. Similarly, a fourth student enters the room, goes up to every fourth locker, opens the closed ones and closes the open ones. This continues until the 100th student has had his go at the lockers. When all the students are done, which lockers are open?

Answer: 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100.

Experimenting, it is not hard to discover the pattern of which lockers will wind up open. To prove the observed pattern is correct, consider the locker with number *N*. Then that locker will be opened or closed one time for each factor of *N*. (By "factor", we mean *all* numbers that divide evenly into *N*, not just prime numbers, and we're including 1 and *N* as factors).

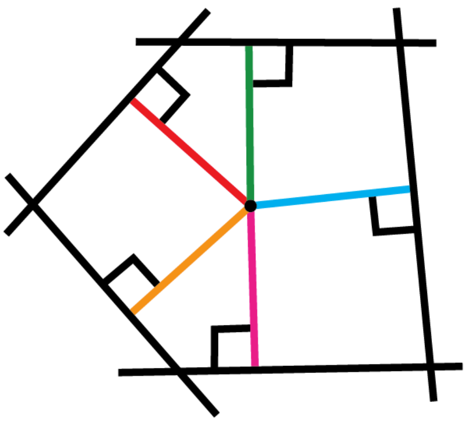
The lockers begin closed. So, if *N* has an even number of factors then it will end up closed. And in fact that will be most lockers. The critical observation is that if *a* is a factor of *N* then so is *N*/*a* (because *a* x *N*/*a* = *N*): so, factors of *N* typically come in pairs.

The only situation where the factors of *N* don't come in pairs occurs when *N* is a square number: in that case, $\sqrt{N}$ is a factor of *N*, and $N/\sqrt{N}$ is just the same factor again.

So, squares numbers are the only ones with an odd number of factors. It follows that it is exactly the square-numbered lockers that will be open at the end.

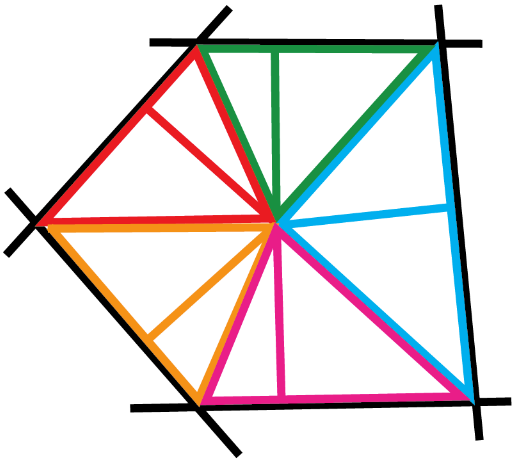
HARD 5

A pentagon is constructed in the following manner: five straight lines of length 4 centimeters are drawn from a point, and then perpendicular lines are drawn to these five lines. It turns out that the pentagon has a perimeter of 40 centimeters. What is its area?



Answer: 80 square centimetres.

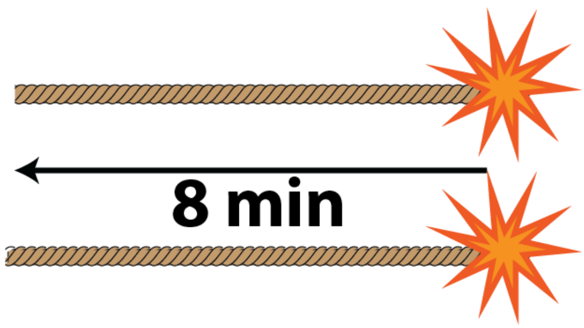
The area of the pentagon is equal to the sum of the areas of the ten right-angled triangles pictured below.



The area of each triangle can be calculated by the familiar 1/2 x base x height. The triangles all have the same height of 4 centimetres. So, we just have to sum the length of the bases: that sum is the perimeter of the pentagon, 40 centimetres. Then, the area of the pentagon is 1/2 x perimeter x height, which works out to 80 cm^2.

HARD 6

You are given two fuses. Each fuse burns for a total of 8 minutes, but they don't burn evenly along their length. How can you use the fuses to measure a time of 6 minutes?



Answer: Simultaneously light two ends of one fuse, and one end of the other.

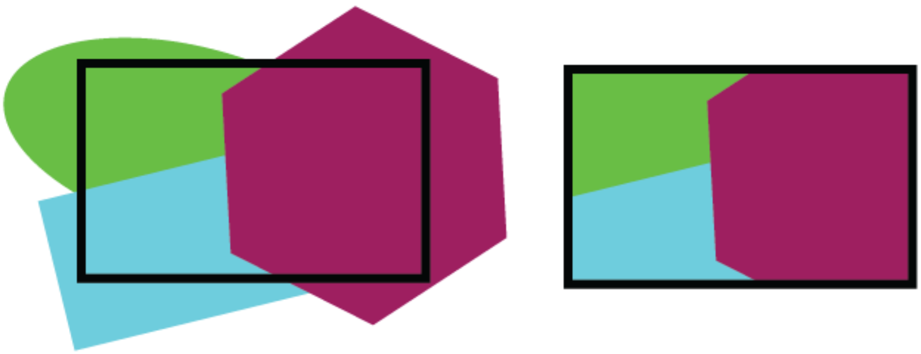
Simultaneously light both ends of one fuse and one end of the other. The first fuse will take 4 minutes to fizzle out, at which time the second fuse will have 4 minutes to burn. At that moment, light the second end of the second fuse. The second fuse will burn out after a further 2 minutes, amounting to 6 minutes in total.

HARD 7

15 irregular paper shapes are used to completely cover the top of a table. The shapes may overlap and may hang over the edge of the table. Show that you can remove five of the paper shapes so that the remaining ten shapes cover at least 2/3 of the table.

Answer: Think of the 15 shapes as three groups of 5 shapes.

First imagine that, instead of 15 shapes, we have just 3 shapes covering the table. Now, cut off any overlaps, so that the paper shapes just cover the table, as pictured.



Then the smallest of the 3 trimmed pieces will have area at most 1/3 the area of the table (otherwise the total area of the 3 shapes would sum to more than that of the table). So, the other 2 pieces must cover at least 2/3 of the table, and the same would have been true before we removed the overlaps.

Now, with 15 shapes covering the table, just imagine them taped to together in groups of 5, creating 3 mega-shapes. Then we know that 2 of these mega-shapes cover at least 2/3 of the table, and so the underlying 10 shapes do so as well.

HARD 8

You are blindfolded and led to a table, upon which are laying 112 coins. 38 of the coins are showing heads, with the rest being tails. How can you separate the coins into two groups, so that each group has the same number of heads? You may move the coins, but you're not allowed to feel them to determine which side is up.

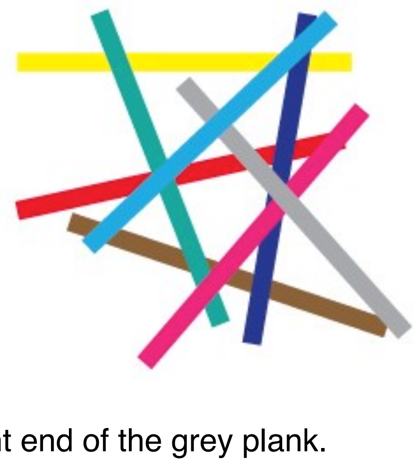
Answer: Group any 38 of the coins together, and turn them over.

Suppose there are *N* heads among the 38 coins you selected. That means the other 38 – *N* coins in that group are tails. So, when turn over the coins, there will be 38 – *N* heads in that group.

However, you also know that there were originally 38 heads in total, which means there must have been 38 – *N* heads in the group of coins you didn't select. So, both groups must have the same number (38 – *N*) of heads: we just don't know what that number is.

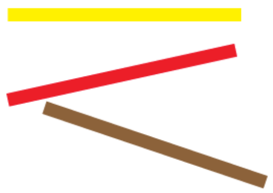
HARD 9

A number of planks are piled on top of each other on a flat floor, as pictured. The planks are all the same size and shape. Which point of the pile is highest above the floor?



Answer: The bottom right end of the grey plank.

The stack has been built as follows. First, the yellow, red and brown planks were placed on the ground.

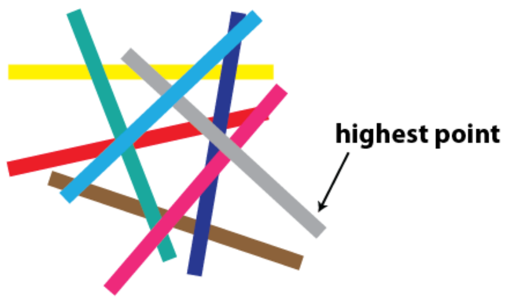


Next, the green and dark blue planks were placed on top.

Then, the pink and light blue planks were set down.

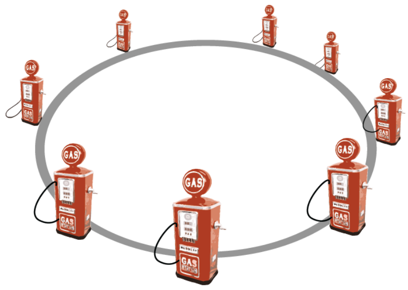
So, at this stage, all the planks are parallel with the ground, and with the pink and light planks on top.

Finally, we slide the gray plank at an angle, so that it rests on the yellow and pink planks. So, the far end of the grey plank will be the highest. (The light blue plank may be wedged up slightly, but not sufficiently to be higher than the high point of the grey plank).



HARD 10

You want to drive around a circular route. You can choose to start at any petrol station along the route, with an empty tank. However, because of a shortage, the petrol stations between them have just enough petrol for you to complete the route. Show that if you choose the beginning station correctly, you can still manage your trip.



Answer: First imagine you have enough fuel in the petrol tank for the whole trip.

Given enough fuel, you could of course start at any petrol station: just pick one. You could still take the petrol on offer at each station, and if you did then you'd end your trip with exactly the same amount of petrol with which you began.

Now, during this imagined trip, there will be a particular petrol station at which your petrol supply would be at a minimum (just before taking the petrol from that station). That's the petrol station from which you can begin your real trip, in which you start with no fuel.

The point is, we know that if you start your trip with L litres of petrol at the chosen station then you'll have at least L litres throughout the trip. The real trip just corresponds to the case where $L = 0$.